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COMMENT

One-dimensional percolation with many particles per site

P M Binder

Center for Nonlinear Studies, MS B258, Los Alamos National Laboratory, Los Alamos, NM 87545, USA
and
Santa Fe Institute, 1120 Canyon Road, Sante Fe, NM 87501, USA

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Abstract. We discuss models where particles are added one by one to sites in the one-dimensional lattice. Each site may contain a finite or infinite number of particles. The percolation density depends on the size of the lattice and maximum occupation number of the sites. We derive site and particle cluster distributions in the case of infinite occupation sites.

In the standard one-dimensional percolation problem (Stauffer 1985), lattice sites are occupied independently with probability P. In order to have an infinite cluster, all sites must be occupied. Therefore, a phase transition occurs at $P_c = 1$.

The purpose of this comment is to introduce less trivial percolation problems in one dimension. A few arguments will be given to predict the behaviour of these models.

The models are as follows: particles are added one by one to randomly chosen sites in a one-dimensional lattice. Two possibilities arise: (a) each node may contain an infinite number of particles (unrestricted problem) or (b) nodes may hold a maximum number K of particles. In the second case, the addition of a particle to a full node is rejected. These models correspond to multiparticle adsorption with particles arriving randomly at the adsorbing sites.

We proceed now to discuss the percolation properties of these models. Two properties are of interest: the percolation density at which a particle falls on the last empty site and the distribution of clusters at any density. We will discuss mainly the first property.

In lattices where sites can hold an infinite number of particles it does not make much sense to talk about density; we should talk rather about occupation numbers. Consider a lattice of N sites to which M particles are added one by one at randomly chosen sites. As in the standard one-dimensional model, the probability of having empty or non-empty sites determines the distribution of site clusters. In this case, the probability of a site having n particles is

$$p_n(M) = \left(\frac{1}{N}\right)^n \left(1 - \frac{1}{N}\right)^{M-n}. \tag{1}$$

As in the standard model, the probability that a site belongs to an s-site cluster is given by

$$\pi_s = s p_0^2 (1 - p_0)^s \tag{2}$$

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where $p_0(M) = (1 - N^{-1})^M$ follows from (1). This differs from the number of s clusters per particle given by Stauffer (1985) by a factor of s. We can also calculate the probability that a given node forms part of a cluster of m particles. This is given by

$$\pi_m = p_0^2 \sum_{r=1}^{m} p_r(M) \sum_{s=1}^{m-r} p_s(M) \sum_{t=1}^{m-r-s} p_t(M) \ldots . \tag{3}$$

As an example, the probability that a site belongs to a four-particle cluster after the addition of M particles is given by $\pi_4 = p_0^2(4p_1^4(M) + 9p_1^2(M)p_2(M) + 4p_1(M)p_3(M) + 2p_2^2(M) + p_4(M))$. The terms are contributions from clusters of 4×1, $2 \times 1 + 2$, $1 + 3$, $2 + 2$ and 4 particles, respectively.

We let a be the number of particles per site at percolation; therefore, $M = aN$. We assume that percolation happens after $N/2$ particles have been added to the lattice when only one empty site is left ($p_0 = 1/N$). By comparing this with (1) we have

$$N^{-1} \sim (1 - N^{-1})^{(a-1/2)N} \sim e^{-(a-1/2)}$$

which readily yields $a \sim \frac{1}{2} + \ln N$ for large N. The expected number of particles per site at percolation is seen to depend on the lattice size. This is not unreasonable, since in order to reach the last few empty lattice sites many particles will fall on previously occupied sites. Table 1 shows numerical results for the occupation number at different lattice sizes. These results agree quite well with the estimate given above.

Table 1. Lattice size, measured occupation number at percolation (1000 configurations for each lattice size) and analytical estimate ($\frac{1}{2} + \ln N$) for the unrestricted model.

N	a (simulations)	$\frac{1}{2} + \ln N$
200	5.9 ± 1.0	5.8
1000	7.5 ± 1.0	7.4
9000	9.6 ± 1.0	9.6

The problem becomes more complicated if only K particles are allowed at each lattice site, with K other than one (the standard problem) or infinity (the unrestricted problem just solved). In the general case it is appropriate to speak of a percolation density, corresponding to the number of particles just needed to cover every site divided by NK. The independent addition of each new particle diminishes with a certain probability the existence of empty nodes. The probability of a site being empty after the addition of M particles is recursively given by

$$p_0(M) = 1 - \frac{1}{N} \sum_{n=1}^{M} \frac{p_0(n-1)}{1 - p_K(n-1)} \tag{4}$$

where $p_i(j)$ is the probability of a node having i particles after j particles have been added to the system, consistently with earlier notation. We have $p_0(0) = 1$, $p_K(0) = 0$. Each term of the sum in (4) represents the probability that, out of the non-full nodes, the nth particle will fall on an empty one. In order to solve for p_K, the probability of a node being full, a system of K equations analogous to (4) must be iterated; alternatively, p_K can be found by solving a rather complicated restricted partitions problem. Either calculation is beyond the scope of this comment.

With the caveat that the occupation number depends strongly on the lattice size N, we have performed illustrative simulations of the percolation process for $N = 1000$, averaging over 1000 configurations, for values of K ranging from 2 to 256. These appear in table 2.

Table 2. Maximum number of particles per site, percolation density ($N = 1000$) averaged over 1000 configurations and occupation number per site at percolation.

K	P_c (simulations)	$a = KP_c$
2	0.996 ± 0.003	1.98
4	0.972 ± 0.02	3.89
8	0.817 ± 0.07	6.54
16	0.470 ± 0.08	7.51
32	0.235 ± 0.04	7.53
64	0.118 ± 0.02	7.53
128	0.059 ± 0.01	7.53
256	0.029 ± 0.005	7.53

It is clear that for large enough K the behaviour of these models should be similar to that of the unrestricted model, i.e. if $p_K \ll 1$ then (4) reduces to (1). This is seen in table 2 for $K \geqslant 16$, for which the occupation number is practically the same as in the unrestricted problem (table 1). In these cases, the percolation density is given by $P_c \sim (1/2K) + (1/K) \ln N$. Because of the restriction at small K, the occupation number must necessarily be lower than in the unrestricted case. This is also seen in table 2. On the other hand, noting that $p_K(n) = 1 - (n/N)$ for $K = 1$ allows us to recover from (4) the standard problem results.

Expressions for the occupation number at percolation and the particle and site-cluster distributions in a one-dimensional lattice of infinitely deep sites have been derived in this comment. The occupation number grows as the logarithm of the number of sites in the lattice. We have shown numerically that if only K particles are allowed in each site the occupation number is almost the same as in the unrestricted case for large enough K, and that the percolation density goes as the inverse of the number of particles allowed per site. The $1 < K < \frac{1}{2} + \ln N$ problem remains to be solved. In the near future we plan to address the effects of different dynamics (e.g. a particle arriving to a full site is sent to the nearest non-full neighbour) in the site and particle cluster distributions.

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Reference

Stauffer D 1985 *Introduction to percolation theory* (London: Taylor and Francis)